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Comments on MHV tree amplitudes for conformal supergravitons topological b-model

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Abstract: We use the twistor-string theory on the B-model of $\mathbf{CP}^{3|4}$ to compute the maximally helicity violating(MHV) tree amplitudes for conformal supergravitons. The correlator of a bilinear in the affine Kac-Moody current(Sugawara stress-energy tensor) can generate these amplitudes. We compare with previous results from open string version of twistor-string theory. We also compute the MHV tree amplitudes for both gravitons and gluons from the correlators between stress-energy tensor and current.

Keywords: Topological Strings, D-branes.

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Contents

1. Introduction

In [1], the D1-brane instanton contribution in the topological B-model of $\mathbf{CP}^{3|4}$ reproduces the perturbative scattering amplitudes in super Yang-Mills theory. For tree level MHV amplitudes, D1-brane instanton is a complex line $\mathbf{CP}^1$ that is a curve of genus zero and degree one. On this curve, there is a $(1,0)$ form current which is represented by the product of two fermion fields. Then the MHV scattering amplitude is an integration of the correlators of the current(motivated by [2]) on D1-brane multiplied by external wavefunctions over the holomorphic measure(superspace measure) on the moduli space of D1-brane.

In the open string version of twistor-string theory [3](for the descriptions on open string version, see also [4, 5]), the MHV tree amplitudes for conformal supergravitons(in addition to gluons) are computed by introducing two different types of vertex operators of conformal dimension 1 where the basic operator product expansion(OPE) behaves a single contraction. Each of them describes twistor wavefunction which is homogeneous in twistor coordinates of weight 1 and -1 respectively. The scattering amplitude is obtained by evaluating the expectation value of the product of the vertex operator. The vertex operator consists of the above wavefunction and a differential operator of bosonic twistor variables. When this differential operator acts on the wavefunction containing the exponential, a single contraction appears in the amplitude. One also sees the inner product between the spinors of negative helicity when one takes the product of the wavefunction and the differential operator acting on other wavefunction.

It is natural to ask whether the MHV tree amplitudes for conformal supergravitons can be described in terms of some correlators on D1-brane multiplied by external wavefunctions along the line of [1, 2]? Since the helicity of bottom component of vertex operator describing the graviton is equal to 2(recall that the helicity of bottom component of vertex operator describing the gluon is 1), one can think of the quadratic expression of current on D1-brane we introduced above as one element of the vertex operator at each external particle. That is, so-called Sugawara construction for stress-energy tensor. For the wavefunction, the explicit form is already given in [3]. There exists also prefactor which depends on the spinor of positive chirality(parametrizing the $\mathbf{CP}^1$). Then the vertex operator consists of the wavefunction with a differential operator with respect to bosonic twistor variable, correlator for stress-energy tensor and the prefactor. We will compare our MHV tree amplitudes for three and four-point functions with previous results [3] from open string version of twistor-string theory. We will also compute the MHV amplitudes(three and four-point functions) for gravitons plus gluons from the correlators between stress-energy tensor and the current.

In section 2.1, we describe the plane wavefunction of a massless Yang-Mills particle and introduce a current which has a short-distance behavior. Then by multiplying each wavefunction with a current, the D1-brane instanton contribution to N-gluons tree level amplitude is given. This subsection is a review of [1, 3].

In section 2.2, we consider the plane wavefunction of a massless supergraviton and introduce a stress-energy tensor which is a quadratic expression of the current. The MHV amplitudes for three gravitons and four gravitons are given and compare our results with [3].

In section 2.3, we describe the MHV amplitudes including the gluons for three-point and four-point from the correlator between stress-energy tensor and a current.

In section 3, after discussing the five-point MHV amplitudes briefly, a generalization for arbitrary N-point MHV amplitudes is considered. Other remarks are given.

For the relevant descriptions for gravitons which are not conformal, refer to [6 − 12]. There are also some works [13, 14] in different context related to the open string version of twistor-string theory [4, 5, 3]. There are many relevant works on the MHV tree amplitudes and we just list some of them here [15].

2. MHV tree amplitudes of the B-model of $\mathbf{CP}^{3|4}$

2.1 MHV tree amplitude for gluons

In this subsection, we review the results of [1, 3] on MHV tree amplitude for N gluons. The twistor space wavefunction, corresponding to plane waves where translations can be diagonalized, of a massless Yang-Mills particle with definite momentum $p_{a\dot{a}} = \pi_a \widetilde{\pi}_{\dot{a}}$ takes the form [3]

$$\phi(\lambda, \mu, \psi) = \left(\frac{\lambda}{\pi}\right) \delta(\langle \lambda, \pi \rangle) \exp\left(i \left(\frac{\pi}{\lambda}\right) [\mu, \widetilde{\pi}]\right) u\left(\left(\frac{\pi}{\lambda}\right) \psi\right) \tag{2.1}$$

which is homogeneous in twistor coordinates $Z^I = (\lambda, \mu, \psi)$ of weight 0 due to the well-defined ratio $\pi/\lambda (= \pi^1/\lambda^1)$ and $u\left((\pi/\lambda)\,\psi\right)$ is the fermionic wavefunction that determines helicity state. Under $(\pi, \widetilde{\pi}) \to (t\pi, t^{-1}\widetilde{\pi})$, the $\psi = 0$ component of ϕ scales as t^{-2} implying that this operator represents a state in Minkowski spacetime of helicity 1(represented by a vertex operator that scales as t^{-2} also). Note that the delta function $\delta(\langle \lambda, \pi \rangle)$ is homogeneous of degree -1 in both λ and π [16].

The current $J^r = \alpha T^r \beta$, where $r = 1, 2, \ldots, \mathrm{dim}G$ and T^r is a generator in the fundamental representation of the group, has a degree of homogeneity in π equal to -2 [2, 3]. It is clear that the product of ϕ and J^r scales as t^{-2}. The scattering amplitude obtained by calculating the expectation value of the product of vertex operators will scale as t^{-2} also. Here α and β denote two dimensional free fermions defined on the $\mathbf{CP}^1$, that is a curve in twistor space, with homogeneous coordinates π_a. The short-distance operator product expansion(OPE) between the two currents is given by [17, 18]

$$J^{r_1}(z_1)J^{r_2}(z_2) = \frac{k\delta^{r_1r_2}}{z_{12}^2} + \frac{f^{r_1r_2r_3}J^{r_3}(z_2)}{z_{12}} + \cdots \tag{2.2}$$

where k is a central charge or level, $f^{r_1r_2r_3}$ is a structure constant, the dots are an infinite set of other regular terms and $z_{12} \equiv z_1 - z_2$. The construction of free fermion fields α and β leads to a central charge $k = 1$ from their defining OPE's $\beta(z_1)\alpha(z_2) = \frac{1}{z_{12}} + \cdots$ [2] by plugging $J^{r_i} = \alpha T^{r_i}\beta$ into (2.2) and computing the OPE. By acting the vacuum state on the above OPE (2.2), the two-point function is given by

$$\langle J^{r_1}(z_1)J^{r_2}(z_2)\rangle = \frac{k\delta^{r_1r_2}}{z_{12}^2}$$

and three-point function can be obtained from (2.2) similarly

$$\langle J^{r_1}(z_1)J^{r_2}(z_2)J^{r_3}(z_3)\rangle = \frac{kf^{r_1r_2r_3}}{z_{12}z_{23}z_{31}}\,. \tag{2.3}$$

Moreover, when we compute higher correlation function, due to the double contractions between the current algebra, there exist multi-trace contributions in the amplitudes.[1]

Using the homogeneous coordinates (λ, μ, ψ) of $\mathbf{CP}^{3|4}$ to set $\lambda_i^1 = 1$ for all i. A degree 1 instanton has $z_i = \lambda_i^2$.[2] Then there exist following relations [3]

$$\frac{\lambda_k}{\pi_k} = \frac{\lambda_k^1}{\pi_k^1} = \frac{1}{\pi_k^1}\,, \qquad z_{jk} = z_j - z_k = -\frac{\langle \pi_j, \pi_k\rangle}{\pi_j^1\pi_k^1} = \frac{\langle j, k\rangle}{\pi_j^1\pi_k^1}\,. \tag{2.4}$$

In the second relation, we already use the delta function constraint: the vertex operator for the j^{th} external particle has a delta function $\delta(\langle\lambda_j, \pi_j\rangle) = \delta(\pi_j^2 - z_j\pi_j^1)$. Therefore, each delta function leads to a factor (λ_j/π_j) due to the following integral

$$\int dz_j \delta(\pi_j^2 - z_j\pi_j^1) = \left(\frac{\lambda_j}{\pi_j}\right)\,. \tag{2.5}$$

The left hand side has a homogeneity $2 - 1 = 1$ in λ_j because $dz_j = \langle\lambda_j, d\lambda_j\rangle$ which has a degree 2 and the delta function has a degree -1. This is consistent with the homogeneity for the right hand side.

[1] For example, one can compute the four-point function explicitly $\langle J^{r_1}(z_1)J^{r_2}(z_2)J^{r_3}(z_3)J^{r_4}(z_4)\rangle = k^2\left(\frac{\delta^{r_1r_2}\delta^{r_3r_4}}{z_{12}^2z_{34}^2} + (2\leftrightarrow 3) + (2\leftrightarrow 4)\right) + k\left(\frac{f^{r_1r_2r_5}f^{r_3r_4r_5}}{z_{12}z_{23}z_{24}z_{34}} + (2\leftrightarrow 3) + (2\leftrightarrow 4)\right)$ where the symbol $(i \leftrightarrow j)$ means that we simply exchange z_i and z_j(and r_i and r_j) for each explicit quantity. The disconnected part of the current correlator(for example, $k^2\frac{\delta^{r_1r_2}\delta^{r_3r_4}}{z_{12}^2z_{34}^2}$ term) leads to a double contraction in both $z_1 - z_2$ and $z_3 - z_4$ channels [1]. By writing $z_{24} = z_{21} + z_{14}$ in the first term of linear in k above, one can extract the term $\frac{1}{z_{12}z_{23}z_{34}z_{41}}$ in four-point correlator by ignoring the double contraction of z_{12}^2.

[2] We denote $(\lambda_k, \mu_k, \psi_k) = (\lambda(z_k), \mu(z_k), \psi(z_k))$ and parametrize the image of z_k in twistor space.

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By multiplying each wavefunction $\phi_j(\lambda, \mu, \psi)$ (2.1) characterized by j^{th} external particle with the current J^{r_j} and taking the vacuum expectation values, the D1-brane instanton contribution of degree 1 to N-gluon tree level MHV amplitude is given by [1]

$$\int d^4x d^8\theta \left(\prod_{j=1}^{N} dz_j \left(\frac{\lambda_j}{\pi_j}\right) \delta(\langle \lambda_j, \pi_j \rangle) \exp\left(i \left(\frac{\pi_j}{\lambda_j}\right) [\mu, \widetilde{\pi}_j]\right) u \left(\left(\frac{\pi_j}{\lambda_j}\right) \psi_j\right)\right) \left\langle \prod_{k=1}^{N} J^{r_k}(z_k) \right\rangle$$

where the N-point current correlator for single-trace amplitude can be generalized and written in terms of λ_i and π_i and is given by

$$\left\langle \prod_{k=1}^{N} J^{r_k}(z_k) \right\rangle = \prod_{i=1}^{N} \frac{1}{z_{i(i+1)}} = \prod_{i=1}^{N} \frac{(\pi_i/\lambda_i)^2}{\langle i, i+1 \rangle} \tag{2.6}$$

with a group theory factor and this is homogeneous of degree 0 in each π_i(recall that $i = \pi_i$) and -2 in each λ_i. We used the second relation of (2.4). Now it is easy to see the factor $(\lambda_j/\pi_j)^2$ coming from both the wavefunction and a delta function is canceled out the factor $(\pi_j/\lambda_j)^2$ in the current correlator (2.6). In the integral over $x_{a\dot{a}}$, we use the twistor equation $\mu^{\dot{a}} = x^{a\dot{a}}\lambda_a$ and this will lead to the usual delta function of energy-momentum conservation and the integral over the θ^{Aa} will give us a factor $\langle r, s\rangle^4$ where the particles r and s have negative helicity in their multiplets.

The final expression for MHV amplitude for N gluons in terms of correlation functions of currents on $\mathbf{CP}^1$ is then

$$(2\pi)^4 \delta^4 \left(\sum_{j=1}^{N} p_j^{a\dot{a}} \right) \langle r, s \rangle^4 \prod_{i=1}^{N} \frac{1}{\langle i, i+1 \rangle}.$$

In next subsections, we would like to see the MHV tree amplitude for N external particles including the supergravitons and gluons.

2.2 MHV tree amplitude for supergravitons

The twistor space wavefunction corresponding to plane waves of a massless supergraviton with definite momentum $p_{a\dot{a}} = \pi_a \widetilde{\pi}_{\dot{a}}$ is given by [3]

$$f^{\dot{a}}(\lambda, \mu, \psi) = \left(\frac{\lambda}{\pi}\right)^2 \widetilde{\pi}^{\dot{a}} \delta(\langle \lambda, \pi \rangle) \exp\left(i \left(\frac{\pi}{\lambda}\right) [\mu, \widetilde{\pi}]\right) u \left(\left(\frac{\pi}{\lambda}\right) \psi\right) \tag{2.7}$$

which is homogeneous in twistor coordinates $Z^I = (\lambda, \mu, \psi)$ of weight 1. Under $(\pi, \widetilde{\pi}) \to (t\pi, t^{-1}\widetilde{\pi})$, the $\psi = 0$ component of $f^{\dot{a}}$ scales as t^{-4} implying that this operator represents a state in Minkowski spacetime of helicity 2.

What is vertex operator for graviton? One can think of quadratic expression of the current, $J^r J^r(z)$, which is nothing but stress-energy tensor $T(z)$, has a degree of homogeneity in π equal to -4 because $J^r(z)$ has a degree of homogeneity equal to -2. Obviously, the product of $f^{\dot{a}}$ and $J^r J^r$ scales as t^{-4}. As for gluons, two dimensional free fermions, defined on the $\mathbf{CP}^1$ with homogeneous coordinates π_a, can be constructed. The exact coefficient of $J^r J^r$ in $T(z)$ depends on both the level of affine Kac-Moody algebra and dual Coxeter number of underlying finite dimensional Lie algebra.

Among possible vector field $f^I \partial_I$, only on the vector fields $f^{\dot{a}} \partial_{\dot{a}}$, the translation generators $c^{a\dot{a}} \lambda_a \frac{\partial}{\partial \mu^{\dot{a}}}$, where $c^{a\dot{a}}$ is a constant vector in the spinor notation, can be diagonalized. Recall that in [3], the external particles have the wavefunctions which are plane waves and in this case, translations can be diagonalized. We also restrict ourselves to this particular wavefunctions.

We expect that the scattering amplitudes can be obtained by multiplying each wavefunction(vector field) with stress-energy tensor and taking the vacuum expectation values. Let us describe each case in detail.

Three-point amplitude for three gravitons. In order to compute the amplitude, one has to get an expression for the correlators between stress-energy tensor. It is very well-known that the short-distance OPE between the two stress-energy tensors in two dimensional conformal field theory [19, 20] is given by

$$T(z_1)T(z_2) = \frac{c/2}{z_{12}^4} + \frac{2T(z_2)}{z_{12}^2} + \frac{\partial T(z_2)}{z_{12}} + \cdots \tag{2.8}$$

where c is a central charge and the dots stand for the terms regular in the limit $z_1 \to z_2$ and we use a simplified notation $z_{12} \equiv z_1 - z_2$. The vacuum expectation value of this OPE(two-point correlation function) is written as

$$\langle T(z_1)T(z_2)\rangle = \frac{c/2}{z_{12}^4}.$$

One can also see this fact from the mode expansion for $T(z)$ by requiring the regularity of $T(z)|0\rangle$ at $z = 0$ [20]. This two-point function is invariant under translations, rotations, dilatations and special conformal transformation. The three-point correlation function can be obtained from the above OPE (2.8):

$$\langle T(z_1)T(z_2)T(z_3)\rangle = \frac{c}{z_{12}^2 z_{23}^2 z_{31}^2}\,. \tag{2.9}$$

The three points z_1, z_2, and z_3 can always be mapped to three reference points, say, $\infty, 1$, and 0 by a conformal transformation. This is what we need in order to compute the three particle scattering amplitude. This looks like (2.3), but the double contractions are present.

With this preliminary view, let us, first, consider the subamplitude where each μ-derivative $\frac{\partial}{\partial \mu_{i+1}}$ of vertex operator at each i^{th} particle acts on next $(i+1)^{th}$ particle. The adjoining two spinor indices between $\widetilde{\pi}_i^{\dot{a}}$ from the wavefunction (2.7) and $\mu_{i+1}^{\dot{a}}$ in the μ-derivative are contracted each other. Remember that in the notation $\mu_i^{\dot{a}}$, the subscript i implies the i^{th} particle the μ-derivative acts on and the spinor index $\dot{a}$ implies the vertex operator it belongs to. Since the μ's of the twistor variable in the wavefunctions (2.7) appear only in the exponentials, one can put each μ-derivative $\frac{\partial}{\partial \mu_{i+1}}$ in front of corresponding each $(i+1)^{th}$ exponential. We introduce a prefactor $\left(\frac{\pi_i^a \pi_{ib}}{\pi_i^1 \pi_i^1}\right)$, which is obviously homogeneous of degree 0 in π_i(does not spoil the behavior of vertex operator by scales under π and

$\widetilde{\pi}$) and degree 2 in λ_i by recalling (2.4), in the vertex operator at the i^{th} particle. Each spinor index at i^{th} particle in this prefactor is contracted with both the one at $(i+1)^{th}$ particle and the one at $(i-1)^{th}$ particle. The role of this factor, on the one hand, makes the scattering amplitude to be homogeneous of degree 0 in each λ_i. Even if the individual degree of each λ_i does not vanish, one can find the appropriate prefactors make the sum of degrees of λ_i's vanish. We will see this in next example. On the other hand, it kills the factor $z_{i(i+1)}$ in the correlator (2.9).

The vertex operator-integrand for this particular case is given by the prefactors, correlator, and the wavefunctions: $\left(\frac{\pi_1^a \pi_{1c}}{\pi_1^1 \pi_1^1}\right)\left(\frac{\pi_2^b \pi_{2a}}{\pi_2^1 \pi_2^1}\right)\left(\frac{\pi_3^c \pi_{3b}}{\pi_3^1 \pi_3^1}\right)\left\langle \prod_{k=1}^3 T(z_k)\right\rangle f_1^{\dot a}\frac{\partial}{\partial \mu_2^{\dot a}} f_2^{\dot b}\frac{\partial}{\partial \mu_3^{\dot b}} f_3^{\dot c}\frac{\partial}{\partial \mu_1^{\dot c}}$. Let us write this vertex operator explicitly in order to compute the subamplitude as follows:

$$\begin{aligned}\int d^4x d^8\theta dz_1 dz_2 dz_3 \ & \left(\frac{\pi_1^a \pi_{1c}}{\pi_1^1 \pi_1^1}\right)\delta(\langle \lambda_1,\pi_1\rangle)\left(\frac{\lambda_1}{\pi_1}\right)^2 \widetilde{\pi}_1^{\dot a}\frac{\partial}{\partial \mu_2^{\dot a}} \exp\left(i\left(\frac{\pi_2}{\lambda_2}\right)[\mu_2,\widetilde{\pi}_2]\right)\times \\ & \times\left(\frac{\pi_2^b \pi_{2a}}{\pi_2^1 \pi_2^1}\right)\delta(\langle \lambda_2,\pi_2\rangle)\left(\frac{\lambda_2}{\pi_2}\right)^2 \widetilde{\pi}_2^{\dot b}\frac{\partial}{\partial \mu_3^{\dot b}} \exp\left(i\left(\frac{\pi_3}{\lambda_3}\right)[\mu_3,\widetilde{\pi}_3]\right)\times \\ & \times\left(\frac{\pi_3^c \pi_{3b}}{\pi_3^1 \pi_3^1}\right)\delta(\langle \lambda_3,\pi_3\rangle)\left(\frac{\lambda_3}{\pi_3}\right)^2 \widetilde{\pi}_3^{\dot c}\frac{\partial}{\partial \mu_1^{\dot c}} \exp\left(i\left(\frac{\pi_1}{\lambda_1}\right)[\mu_1,\widetilde{\pi}_1]\right)\times \\ & \times\left\langle \prod_{k=1}^3 T(z_k)\right\rangle \prod_{j=1}^3 u\left(\left(\frac{\pi_j}{\lambda_j}\right)\psi_j\right). \end{aligned} \tag{2.10}$$

Let us simplify this by performing z_i integral first. After differentiating the exponentials with respect to each μ's, the factor $[1,2][2,3][3,1]$ for the spinors of negative chirality arises in this amplitude together with a factor $i^3\prod_{i=1}^3(\pi_i/\lambda_i)$. Using the second relation of (2.4)(in other words, by changing the prefactors into $z_{i(i+1)}$'s), the factor $z_{12}z_{23}z_{31}$ in the denominator of the correlator $\langle T(z_1)T(z_2)T(z_3)\rangle$, characterized by (2.9), is canceled out. As a result, the remaining term leads to the factor $\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle$ for the spinors of positive chirality in the denominator of this amplitude. An extra factor (λ_i/π_i) appears by computing the delta function integration (2.5). As for gluons, for the integral over $x_{a\dot a}$, we use the twistor equation $\mu^{\dot a} = x^{a\dot a}\lambda_a$ and this will lead to the usual delta function of energy-momentum conservation and the integral over the θ^{Aa} will give us a factor $\langle r,s\rangle^4$ where particles r and s have the minimum helicity 0 in their multiplets. Note that the top component of f-type vertex operator [3] provides this scalar $\overline{C}$ of helicity 0 while the bottom component of f-type vertex operator gives a graviton e_2 of maximum helicity 2.

Finally, the subamplitude we get, after taking account of all the numerical factors correctly, is given by

$$(-i)^3\langle r,s\rangle^4\frac{[1,2][2,3][3,1]}{\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle}. \tag{2.11}$$

Of course, after we assign the particles r and s, this can be further simplified.

Similarly, for the subamplitude where μ-derivative of vertex operator at 1^{st} particle acts on the wavefunction of 3^{rd} particle, μ-derivative of vertex operator at 3^{rd} particle acts on the wavefunction of 2^{nd} particle, and finally μ-derivative of vertex operator at 2^{nd}

particle acts on the wavefunction of 1^{st} particle(or equivalently by interchanging the roles of 2^{nd} particle and 3^{rd} particle:$(2 \leftrightarrow 3)$), one gets, by following above procedure,

$$(-i)^3 \langle r, s\rangle^4 \frac{[1,3][2,1][3,2]}{\langle 1,3\rangle\langle 2,1\rangle\langle 3,2\rangle} \tag{2.12}$$

with a little rearrangement.

Next, let us consider the subamplitude where the μ-derivatives of vertex operator at both 1^{st} particle and 3^{rd} particle act on the wavefunction of 2^{nd} particle simultaneously. Moreover, the μ-derivative of vertex operator at 2^{nd} particle acts on the wavefunction of 3^{rd} particle. In this case, since there exists only one spinor of negative helicity $\widetilde{\pi}_1^{\dot{a}}$ at 1^{st} particle, compared with previous cases, the cubic in π_1^a(the degree of λ_1 is three) of the vertex operator at 1^{st} particle appears as a prefactor. Similarly, for the 2^{nd} particle, only linear term in π_2^a(the degree of λ_2 is one) of the vertex operator at 2^{nd} particle arises as a prefactor because there is a cubic in $\widetilde{\pi}_2^{\dot{a}}$ after differentiating the exponentials. For the 3^{rd} particle, the prefactor is the same as before: quadratic in π_3^a(the degree of λ_3 is two). The corresponding spinor indices in these prefactors are contracted properly between these particles. Note that although the homogeneity in both λ_1 and λ_2 is not equal to zero in the amplitude(due to the different prefactors which are not quadratic in π_i^a and there is no appearance for μ_1 derivative), compared with previous cases, the sum of degrees is zero. With this information, one can write the subamplitude as

$$\begin{aligned}\int d^4x d^8\theta dz_1 dz_2 dz_3 & \left(\frac{\pi_1^a \pi_{1b}\pi_{1c}}{\pi_1^1\pi_1^1\pi_1^1}\right)\delta(\langle\lambda_1,\pi_1\rangle)\left(\frac{\lambda_1}{\pi_1}\right)^2 \widetilde{\pi}_1^{\dot{a}} \frac{\partial}{\partial \mu_2^{\dot{a}}}\frac{\partial}{\partial \mu_2^{\dot{c}}} \exp\left(i\left(\frac{\pi_2}{\lambda_2}\right)[\mu_2,\widetilde{\pi}_2]\right) \times \\ & \times \left(\frac{\pi_{2a}}{\pi_2^1}\right)\delta(\langle\lambda_2,\pi_2\rangle)\left(\frac{\lambda_2}{\pi_2}\right)^2 \widetilde{\pi}_2^{\dot{b}} \frac{\partial}{\partial \mu_3^{\dot{b}}} \exp\left(i\left(\frac{\pi_3}{\lambda_3}\right)[\mu_3,\widetilde{\pi}_3]\right) \times \\ & \times \left(\frac{\pi_3^b \pi_3^c}{\pi_3^1\pi_3^1}\right)\delta(\langle\lambda_3,\pi_3\rangle)\left(\frac{\lambda_3}{\pi_3}\right)^2 \widetilde{\pi}_3^{\dot{c}} \exp\left(i\left(\frac{\pi_1}{\lambda_1}\right)[\mu_1,\widetilde{\pi}_1]\right) \times \\ & \times \left\langle \prod_{k=1}^{3} T(z_k)\right\rangle \prod_{j=1}^{3} u\left(\left(\frac{\pi_j}{\lambda_j}\right)\psi_j\right).\end{aligned}$$

Then, the factor $[1,2][2,3][3,2]$ for the spinors of negative chirality arises in this amplitude by keeping track of both spinor index in $\widetilde{\pi}^{\dot{a}}$ and particle index in μ_j appropriately. Using the second relation of (2.4), the factor $z_{12}z_{31}z_{31}$ in the denominator of the correlator $\langle T(z_1)T(z_2)T(z_3)\rangle$ is removed. As a result, only $\frac{1}{z_{12}z_{23}^2}$ survives in this correlator. We expect that there should be $1/(\pi_1^1)^2$ since in the prefactor there is an extra factor $1/\pi_1^1$ and there is no μ_1-derivative implying another extra factor $1/\pi_1^1$. Similarly, $(\pi_2^1)^2$ will appear in the subamplitude because the effects from extra μ_2-derivative and a single prefactor will add this factor. Finally, by simplifying this, the subamplitude can be written as

$$(-i)^3 \langle r, s\rangle^4 \frac{[1,2][2,3][3,2]\langle 2,\zeta\rangle^2}{\langle 1,2\rangle\langle 2,3\rangle\langle 3,2\rangle\langle 1,\zeta\rangle^2} \tag{2.13}$$

where a spinor $\zeta^a = (0,1)$ is introduced [3]: $(\pi_1^1)^2 = \langle 1,\zeta\rangle^2$ and $(\pi_2^1)^2 = \langle 2,\zeta\rangle^2$. One can count the homogeneity 2 of λ_1 for 1^{st} particle above amplitude as follows: the integration measure dz_1 has 2 as in the discussion of (2.5), the prefactor has 3, delta function has -1, the scale factor has 2, and the correlator has -4 from (2.9) of homogeneity in λ_1. This reflects on the final result (2.13) where there is a factor $(\pi_1^1)^2$ in the denominator by remembering (2.4). Similarly, one can count the homogeneity of λ_2 for the 2^{nd} particle which is equal to -2. This is reminiscent of a factor $(\pi_2^1)^2$ in the numerator of (2.13). Here the sum of degree of λ_1 and degree of λ_2 vanishes.

One can analyze the other cases by permutations between three particles. In fact, there are other five cases coming from the permutations between 1, 2 and 3: (132), (312), (213), (231), and (321). Then the remaining scattering amplitude can be written as

$$(-i)^3\langle r,s\rangle^4\left(\frac{[1,3][2,3][3,2]\langle 3,\zeta\rangle^2}{\langle 1,3\rangle\langle 2,3\rangle\langle 3,2\rangle\langle 1,\zeta\rangle^2}+\text{four other terms}\right) \tag{2.14}$$

where the first term is obtained by interchanging the index 2 and index 3 from (2.13) with a rearrangement. Now it is ready to collect all the subamplitudes and sum over them. Combining (2.11), (2.12), (2.13) and five other cases characterized by (2.14)(obtained by permutations), one writes the full three particle scattering amplitude consisting of eight terms($2^3=8$) as a simplified and symmetric form

$$(-i)^3\langle r,s\rangle^4\prod_{j=1}^{3}\sum_{k(\neq j)=1}^{3}\frac{[j,k]\langle k,\zeta\rangle^2}{\langle j,k\rangle\langle j,\zeta\rangle^2}\,. \tag{2.15}$$

This is exactly an expression for the MHV amplitude [3] for scattering of three particles: a graviton of helicity 2 and two particles labeled by r and s which are scalars $\overline{C}$ of helicity 0.

So far, we have constructed the vertex operator at each particle and computed the amplitudes by executing the integrals. In doing this, we have to introduce a prefactor for each particle which is a function of π_i^a as well as both the correlator of stress-energy tensor and the wavefunction of plane wave. The determination for this prefactor can be understood from the homogeneity of the sum of degrees for λ_i's. Can we find N particle scattering amplitude? We will describe four particle case next.

Four-point amplitude for four gravitons. Now let us consider four graviton scattering amplitude. One can compute four-point function from the results for three-point function(and their derivatives) and two-point function we have obtained(We will not present all the details here. Refer to [21] for the details). It turns out to be

$$\begin{aligned}\langle T(z_1)T(z_2)T(z_3)T(z_4)\rangle &= c^2\left(\frac{1}{z_{12}^4z_{34}^4}+\frac{1}{z_{13}^4z_{24}^4}+\frac{1}{z_{14}^4z_{23}^4}\right)+\\ &\quad+\frac{2c}{z_{23}^2z_{34}^2z_{24}^2}\left(\frac{1}{z_{12}^2}+\frac{1}{z_{13}^2}+\frac{1}{z_{14}^2}-\frac{1}{z_{12}z_{14}}-\frac{1}{z_{13}z_{14}}-\frac{1}{z_{12}z_{13}}\right)\\ &=\frac{1}{z_{13}^4z_{24}^4}\left[c^2\left(1+\frac{1}{x^4}+\frac{1}{(1-x)^4}\right)+2c\frac{(1-x+x^2)}{x^2(1-x)^2}\right]\\ &\equiv\frac{1}{z_{13}^4z_{24}^4}G\begin{pmatrix}2&1\\3&4\end{pmatrix}(x)\end{aligned} \tag{2.16}$$

where $z_{ij} \equiv z_i - z_j$, $x \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}}$ and $1-x \equiv \frac{z_{14}z_{23}}{z_{13}z_{24}}$. The quadratic terms in c come from the sum of products of two-point functions $\langle T(z_i)T(z_j)\rangle\langle T(z_k)T(z_l)\rangle$: disconnected piece. Note that due to the overall factor for linear term in c

$$\frac{1}{z_{13}^4 z_{24}^4}\frac{1}{x^2(1-x)^2} = \frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{41}^2},$$

the linear term of c in four-point function is given by two terms:

$$\frac{2c}{z_{12}z_{23}z_{34}z_{41}}\left(\frac{1}{z_{12}z_{23}z_{34}z_{41}} + \frac{1}{z_{13}^2 z_{24}^2}\right) \rightarrow \frac{2c}{z_{12}^2 z_{23}^2 z_{34}^2 z_{41}^2} \tag{2.17}$$

where the first term behaves like the one given in three point function.[3] Even at the linear level for c, there exist two different contributions. We only consider the contribution from the first term above when we insert the correlator for stress-energy tensor into the scattering amplitude. Recall that for current correlator, single-trace part is proportional to k while multi-trace part is proportional to higher powers of k. The crossing symmetry conditions(for example, [19, 20, 22]) that correspond to a change of channels are characterized by

$$\begin{aligned} G\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}(x) &= (-1)^{h_1+h_2+h_3+h_4} x^{-2h_3} G\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}\left(\frac{1}{x}\right), \\ G\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}(x) &= (-1)^{h_1+h_2+h_3+h_4} G\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}(1-x) \end{aligned} \tag{2.18}$$

where h_i is a conformal dimension for $T(z_i)$. In this case, $h_i = 2$ where $i = 1, \ldots, 4$. It can be easily checked that the conditions (2.18) for crossing symmetry with explicit form (2.16) are indeed satisfied.

Let us first consider the subamplitude where the μ-derivative $\partial/\partial\mu_2$ of vertex operator at 1^{st} particle acts on the wavefunction of 2^{nd} particle and the μ-derivative $\partial/\partial\mu_1$ of vertex operator at 2^{nd} particle acts on the wavefunction of 1^{st} particle. The adjoining two spinor indices $\dot{a}$ appearing them(and $\dot{b}$) are contracted each other. Moreover, the μ-derivative of vertex operator at 3^{rd} particle acts on the wavefunction of 4^{th} particle and vice versa. As we have done for three-point amplitude, let us introduce a prefactor $\left(\frac{\pi_i^a \pi_i^b}{\pi_i^1 \pi_i^1}\right)$, which is obviously homogeneous of degree 0 in π, in the vertex operator at the i^{th} particle(Note that there exists a quadratic term $\widetilde{\pi}_i^{\dot{a}}\widetilde{\pi}_i^{\dot{b}}$ for each i^{th} particle after differentiating the exponentials). As we have done before, the integrand of vertex operator contains $f_1^{\dot{a}}\frac{\partial}{\partial\mu_2^{\dot{a}}} f_2^{\dot{b}}\frac{\partial}{\partial\mu_1^{\dot{b}}} f_3^{\dot{c}}\frac{\partial}{\partial\mu_4^{\dot{c}}} f_4^{\dot{d}}\frac{\partial}{\partial\mu_3^{\dot{d}}}$ as well as both the correlator and prefactors. Let us write the amplitude explicitly by substituting the form of the wavefunctions (2.7) as follows:

$$\left(\frac{\pi_{1c}\pi_{1d}}{\pi_1^1\pi_1^1}\right)\left(\frac{\pi_2^a\pi_2^b}{\pi_2^1\pi_2^1}\right)\left(\frac{\pi_{3a}\pi_{3b}}{\pi_3^1\pi_3^1}\right)\left(\frac{\pi_4^c\pi_4^d}{\pi_4^1\pi_4^1}\right)\times$$

[3] One can write $z_{13}z_{24}$ in terms of other variables. That is, $z_{13}z_{24} = z_{12}z_{34} - z_{23}z_{41}$. Then the second term of (2.17) reduces to the first term of (2.17) if we ignore double contractions $z_{i(i+1)}^2$ where $i = 1, \ldots, 4$.

$$\times \left\langle \prod_{k=1}^{4} T(z_k) \right\rangle \left(\prod_{j=1}^{4} (\lambda_j/\pi_j)^2 \, \delta(\langle \lambda_j, \pi_j \rangle) u \left((\pi_j/\lambda_j) \, \psi_j \right) \right) \times$$
$$\times \widetilde{\pi}_1^{\dot{a}} \frac{\partial}{\partial \mu_2^{\dot{a}}} \exp \left(i \left(\frac{\pi_2}{\lambda_2} \right) [\mu_2, \widetilde{\pi}_2] \right) \widetilde{\pi}_2^{\dot{b}} \frac{\partial}{\partial \mu_1^{\dot{b}}} \exp \left(i \left(\frac{\pi_1}{\lambda_1} \right) [\mu_1, \widetilde{\pi}_1] \right) \times$$
$$\times \widetilde{\pi}_3^{\dot{c}} \frac{\partial}{\partial \mu_4^{\dot{c}}} \exp \left(i \left(\frac{\pi_4}{\lambda_4} \right) [\mu_4, \widetilde{\pi}_4] \right) \widetilde{\pi}_4^{\dot{d}} \frac{\partial}{\partial \mu_3^{\dot{d}}} \exp \left(i \left(\frac{\pi_3}{\lambda_3} \right) [\mu_3, \widetilde{\pi}_3] \right).$$

After differentiating the exponentials with respect to each μ's, the factor $[1,2][2,1][3,4][4,3]$ for the spinors of negative chirality arises in this amplitude.[4] Using the second relation of (2.4), the factor $z_{23}^2 z_{41}^2$ in the denominator of the correlator $\langle \prod_{k=1}^{4} T(z_k) \rangle$, characterized by (2.16) or (2.17), is removed due to the presence of prefactors. Then the remaining factor $\frac{1}{z_{12}^2 z_{34}^2}$, that can be reexpressed as $\frac{1}{\langle 1,2\rangle\langle 2,1\rangle\langle 3,4\rangle\langle 4,3\rangle}$ with some function of π_i^1's, survives in this correlator. It is easy to see that the homogeneity for each λ_i is equal to zero: quadratic prefactors and all the μ_j-derivatives appear once. As a result, the factor $\langle 1,2\rangle\langle 2,1\rangle\langle 3,4\rangle\langle 4,3\rangle$ for the spinors of positive chirality appears in the denominator of this subamplitude. Therefore, it turns out that the subamplitude can be written as

$$(-i)^4 \langle r, s\rangle^4 \frac{[1,2][2,1][3,4][4,3]}{\langle 1,2\rangle\langle 2,1\rangle\langle 3,4\rangle\langle 4,3\rangle}. \tag{2.19}$$

Also, there are two other cases where we simply replace the role of 2^{nd} particle with the one for 3^{rd} particle(can be interpreted as an interchanging of $2 \leftrightarrow 3$) and for second case, we replace 2^{nd} particle with 4^{th} particle($2 \leftrightarrow 4$). That is, the subamplitudes are given by

$$(-i)^4 \langle r, s\rangle^4 \frac{[1,3][2,4][3,1][4,2]}{\langle 1,3\rangle\langle 2,4\rangle\langle 3,1\rangle\langle 4,2\rangle}, \qquad (-i)^4 \langle r, s\rangle^4 \frac{[1,4][2,3][3,2][4,1]}{\langle 1,4\rangle\langle 2,3\rangle\langle 3,2\rangle\langle 4,1\rangle} \tag{2.20}$$

respectively. When we make a permutation between the indices 1,2,3, and 4, there exists $4! = 24$ possibilities. If we apply this permutation to (2.19), then there exist eight of (2.19), eight of first expression of (2.20), and eight of second expression of (2.20).

Next, let us consider the particular amplitude where the twistor derivative with respect to μ acts on $2^{nd}, 3^{rd}, 4^{th}$ and 1^{st} particles successively which is a generalization of (2.10) for four particle scattering and the amplitude can be written as, by simply adding an extra 4^{th} particle,

$$\left(\frac{\pi_1^a \pi_{1d}}{\pi_1^1 \pi_1^1} \right) \left(\frac{\pi_2^b \pi_{2a}}{\pi_2^1 \pi_2^1} \right) \left(\frac{\pi_3^c \pi_{3b}}{\pi_3^1 \pi_3^1} \right) \left(\frac{\pi_4^d \pi_{4c}}{\pi_4^1 \pi_4^1} \right) \left\langle \prod_{k=1}^{4} T(z_k) \right\rangle \times$$
$$\times \prod_{j=1}^{4} \left(\frac{\lambda_j}{\pi_j} \right)^2 \delta(\langle \lambda_j, \pi_j \rangle) u \left(\left(\frac{\pi_j}{\lambda_j} \right) \psi_j \right) \widetilde{\pi}_j^{\dot{a}_j} \frac{\partial}{\partial \mu_{j+1}^{\dot{a}_j}} \exp \left(i \left(\frac{\pi_{j+1}}{\lambda_{j+1}} \right) [\mu_{j+1}, \widetilde{\pi}_{j+1}] \right).$$

The homogeneity for each λ_i is equal to zero: quadratic prefactors and all the μ_j-derivatives appear once. In this case, after writing all the numerical factors correctly, it is easy to see

[4] Here we ignored the integrals over $x_{a\dot{a}}, \theta^{Aa}$ and z_i where $i = 1,2,3,4$ for simplicity.

the result can be written as

$$(-i)^4\langle r,s\rangle^4\frac{[1,2][2,3][3,4][4,1]}{\langle 1,2\rangle\langle 2,3\rangle\langle 3,4\rangle\langle 4,1\rangle}\,. \tag{2.21}$$

Also there are five other cases by changing the role of each particle. In other words, one can easily see that when we make a permutation for the expression (2.21), we get six different combinations including the case (2.21) with multiplicity four.

So far, all the amplitudes we have obtained are homogeneous of degree 0 in each λ_i. That is, there is no ζ dependence in (2.19), (2.20), and (2.21). Now let us consider the other cases where each degree for λ_i is nonzero, in general, but the sum of degrees of λ_i's is equal to zero. Let us describe the subamplitude where the μ-derivatives of vertex operator at both 1^{st} particle and 3^{rd} particle acts on the wavefunction of 2^{nd} particle and the μ-derivatives of vertex operator at both 2^{nd} particle and 4^{th} particle acts on the wavefunction of 1^{st} particle. One can determine the prefactors by requiring the sum of the number of π_i and $\widetilde{\pi}_i$ should be equal to 4. The subamplitude becomes

$$\left(\frac{\pi_{1d}}{\pi_1^1}\right)\left(\frac{\pi_2^a}{\pi_2^1}\right)\left(\frac{\pi_3^b\pi_3^c\pi_{3a}}{\pi_3^1\pi_3^1\pi_3^1}\right)\left(\frac{\pi_4^d\pi_{4b}\pi_{4c}}{\pi_4^1\pi_4^1\pi_4^1}\right)F(z_k,\pi_j,\lambda_j,\psi_j)\times$$
$$\times\widetilde{\pi}_1^{\dot a}\widetilde{\pi}_3^{\dot c}\frac{\partial}{\partial\mu_2^{\dot a}}\frac{\partial}{\partial\mu_2^{\dot c}}\exp\left(i\left(\frac{\pi_2}{\lambda_2}\right)[\mu_2,\widetilde{\pi}_2]\right)\widetilde{\pi}_2^{\dot b}\widetilde{\pi}_4^{\dot d}\frac{\partial}{\partial\mu_1^{\dot b}}\frac{\partial}{\partial\mu_1^{\dot d}}\exp\left(i\left(\frac{\pi_1}{\lambda_1}\right)[\mu_1,\widetilde{\pi}_1]\right)$$

where we introduce a new notation which appears several times later

$$F(z_k,\pi_j,\lambda_j,\psi_j)\equiv\left\langle\prod_{k=1}^4 T(z_k)\right\rangle\prod_{j=1}^4\left(\frac{\lambda_j}{\pi_j}\right)^2\delta(\langle\lambda_j,\pi_j\rangle)u\left(\left(\frac{\pi_j}{\lambda_j}\right)\psi_j\right). \tag{2.22}$$

From above integrand, one can read off the factor $[1,2][2,1][3,2][4,1]$ by keeping track of the structure of spinor index in $\widetilde{\pi}^{\dot a}$ and particle index in μ_j and also the prefactors lead to the fact that the factor $z_{23}z_{34}^2z_{41}$, in the denominator of the correlator for stress-energy tensor, is canceled out. The fact that there are one extra μ_1-derivative, compared with previous case, and linear prefactor for the 1^{st} particle allows us to have an extra $(\pi_1^1)^2$ in the subamplitude. This holds for the 2^{nd} particle. On the other hand, for the 3^{rd} and 4^{th} particles, there are no μ_j-derivative($j=3,4$) and cubic prefactors. This will lead to a factor $1/(\pi_3^1\pi_4^1)^2$ in the amplitude. The final expression for this subamplitude is

$$(-i)^4\langle r,s\rangle^4\frac{[1,2][2,1][3,2][4,1]\langle 1,\zeta\rangle^2\langle 2,\zeta\rangle^2}{\langle 1,2\rangle\langle 2,1\rangle\langle 3,2\rangle\langle 4,1\rangle\langle 3,\zeta\rangle^2\langle 4,\zeta\rangle^2} \tag{2.23}$$

which reflects on the fact that this amplitude is homogeneous of degree -2 in λ_1, -2 in λ_2, 2 in λ_3 and 2 in λ_4. Therefore, the sum of degrees is equal to zero. In this case also, the other cases can be obtained from a permutation on (2.23). Actually, there are twelve different amplitudes including (2.23) with multiplicity 2.

So far, we have only considered the cases where any inner product $[i,j]$ in the subamplitude is a contraction between two consecutive indices i and $j(=i+1)$. From now on, we will describe the subamplitudes for which there is a factor $[i,j]$ where $j=i+2$ as well as the factors $[i,i+1]$.

Let us consider the subamplitude where the μ-derivatives of vertex operator at both 3^{rd} particle and 4^{th} particle act on the wavefunction of 1^{st} particle, the μ-derivative of vertex operator at 1^{st} particle acts on the wavefunction of 4^{th} particle, and the μ-derivative of vertex operator at 2^{nd} particle acts on the wavefunction of 3^{rd} particle. Since the correlator $\left\langle \prod_{k=1}^{4} T(z_k) \right\rangle$ does not contain z_{13} at all, as a prefactor, the term $\left(\frac{\pi_3^a \pi_{1a}}{\pi_3^1 \pi_1^1}\right)^{-1}$ should be present in the amplitude, compared with previous cases. Still effectively, the sum of the number of π_i and $\widetilde{\pi}_i$ should be equal to 4. The quadratic terms in π_i^a for $i = 1, 4$ and cubic terms in π_i^a for $i = 2, 3$ occur. With this in mind, one can write the amplitude with (2.22) as follows:

$$\left(\frac{\pi_1^a \pi_1^b}{\pi_1^1 \pi_1^1}\right)\left(\frac{\pi_2^c \pi_{2a} \pi_{2b}}{\pi_2^1 \pi_2^1 \pi_2^1}\right)\left(\frac{\pi_3^d \pi_3^e \pi_{3c}}{\pi_3^1 \pi_3^1 \pi_3^1}\right)\left(\frac{\pi_{4d} \pi_{4e}}{\pi_4^1 \pi_4^1}\right)\left(\frac{\pi_3^f \pi_{1f}}{\pi_3^1 \pi_1^1}\right)^{-1} \widetilde{\pi}_1^{\dot{a}} \frac{\partial}{\partial \mu_4^{\dot{a}}} \exp\left(i\left(\frac{\pi_4}{\lambda_4}\right)[\mu_4, \widetilde{\pi}_4]\right) \times$$
$$\times \widetilde{\pi}_2^{\dot{b}} \frac{\partial}{\partial \mu_3^{\dot{b}}} \exp\left(i\left(\frac{\pi_3}{\lambda_3}\right)[\mu_3, \widetilde{\pi}_3]\right) \widetilde{\pi}_3^{\dot{c}} \widetilde{\pi}_4^{\dot{d}} \frac{\partial}{\partial \mu_1^{\dot{c}}} \frac{\partial}{\partial \mu_1^{\dot{d}}} \exp\left(i\left(\frac{\pi_1}{\lambda_1}\right)[\mu_1, \widetilde{\pi}_1]\right) F(z_k, \pi_j, \lambda_j, \psi_j).$$

It is easy to see that the only factor $\frac{1}{z_{14} z_{23} z_{41}}$ in the correlator with the help of (2.4) remains because the four prefactors cancel out $z_{12}^2 z_{23} z_{34}^2$. For 3^{rd} particle, the overall factor has $1/(\pi_3^1)^2$ and there is no extra factor for this in the final expression. For 4^{th} particle, the same thing happens. However, for 1^{st} and 2^{nd} particles, the extra π_1^1 and π_2^1 dependence occurs. From this consideration, one can simplify the subamplitude as

$$(-i)^4 \langle r, s\rangle^4 \frac{[1,4][2,3][3,1][4,1]\langle 1, \zeta\rangle^2}{\langle 1,4\rangle\langle 2,3\rangle\langle 3,1\rangle\langle 4,1\rangle\langle 2,\zeta\rangle^2}\,. \tag{2.24}$$

Also due to the permutations between four particles, there are contributions from twenty three other terms.

Let us describe other subamplitude where the μ-derivatives of vertex operator at both 2^{nd} particle and 4^{th} particle acts on the wavefunction of 1^{st} particle, the μ-derivative of vertex operator at 1^{st} particle acts on the wavefunction 3^{rd} particle, and the μ-derivative of vertex operator at 3^{rd} particle acts on the wavefunction of 4^{th} particle. In this case, the presence of factor $\widetilde{\pi}_1^{\dot{a}} \widetilde{\pi}_{3\dot{a}}$ needs to have an extra prefactor in the vertex operator. The π_i^1 dependence in the amplitude is the same as previous case: same prefactors and same μ_j-derivatives. The subamplitude with (2.22) is

$$\left(\frac{\pi_1^a \pi_{1e}}{\pi_1^1 \pi_1^1}\right)\left(\frac{\pi_2^b \pi_2^c \pi_{2a}}{\pi_2^1 \pi_2^1 \pi_2^1}\right)\left(\frac{\pi_3^d \pi_{3b} \pi_{3c}}{\pi_3^1 \pi_3^1 \pi_3^1}\right)\left(\frac{\pi_4^e \pi_{4d}}{\pi_4^1 \pi_4^1}\right)\left(\frac{\pi_3^f \pi_{1f}}{\pi_3^1 \pi_1^1}\right)^{-1} \widetilde{\pi}_1^{\dot{a}} \frac{\partial}{\partial \mu_3^{\dot{a}}} \exp\left(i\left(\frac{\pi_3}{\lambda_3}\right)[\mu_3, \widetilde{\pi}_3]\right) \times$$
$$\times \widetilde{\pi}_3^{\dot{c}} \frac{\partial}{\partial \mu_4^{\dot{c}}} \exp\left(i\left(\frac{\pi_4}{\lambda_4}\right)[\mu_4, \widetilde{\pi}_4]\right) \widetilde{\pi}_2^{\dot{b}} \widetilde{\pi}_4^{\dot{d}} \frac{\partial}{\partial \mu_1^{\dot{b}}} \frac{\partial}{\partial \mu_1^{\dot{d}}} \exp\left(i\left(\frac{\pi_1}{\lambda_1}\right)[\mu_1, \widetilde{\pi}_1]\right) F(z_k, \pi_j, \lambda_j, \psi_j)\,.$$

Therefore, the subamplitude is given by

$$(-i)^4 \langle r, s\rangle^4 \frac{[1,3][2,1][3,4][4,1]\langle 1, \zeta\rangle^2}{\langle 1,3\rangle\langle 2,1\rangle\langle 3,4\rangle\langle 4,1\rangle\langle 2,\zeta\rangle^2}\,. \tag{2.25}$$

By permutations between four particles, there are other contributions from twenty three terms.

Finally, let us describe the subamplitude where the μ-derivatives of vertex operator at $2^{nd}, 3^{rd}, 4^{th}$ particles act on the wavefunction of 1^{st} particle and the μ-derivative of vertex operator at 1^{st} particle acts on the wavefunction of 4^{th} particle. Except 4^{th} particle, there is a dependence on $\pi_i^1 (i = 1, 2, 3)$ in the amplitude. The sum of degree of λ_i's is zero. Then the subamplitude is

$$\left(\frac{\pi_1^a}{\pi_1^1}\right)\left(\frac{\pi_2^b\pi_2^c\pi_{2a}}{\pi_2^1\pi_2^1\pi_2^1}\right)\left(\frac{\pi_3^d\pi_3^e\pi_{3b}\pi_{3c}}{\pi_3^1\pi_3^1\pi_3^1\pi_3^1}\right)\left(\frac{\pi_{4d}\pi_{4e}}{\pi_4^1\pi_4^1}\right)\left(\frac{\pi_3^f\pi_{1f}}{\pi_3^1\pi_1^1}\right)^{-1} F(z_k, \pi_j, \lambda_j, \psi_j) \times$$
$$\times \widetilde{\pi}_1^{\dot{a}} \frac{\partial}{\partial \mu_4^{\dot{a}}} \exp\left(i\left(\frac{\pi_4}{\lambda_4}\right)[\mu_4, \widetilde{\pi}_4]\right) \widetilde{\pi}_2^{\dot{b}}\widetilde{\pi}_3^{\dot{c}}\widetilde{\pi}_4^{\dot{d}} \frac{\partial}{\partial \mu_1^{\dot{b}}}\frac{\partial}{\partial \mu_1^{\dot{c}}}\frac{\partial}{\partial \mu_1^{\dot{d}}} \exp\left(i\left(\frac{\pi_1}{\lambda_1}\right)[\mu_1, \widetilde{\pi}_1]\right).$$

In this case, the subamplitude can be written as

$$(-i)^4 \langle r, s\rangle^4 \frac{[1,4][2,1][3,1][4,1]\langle 1, \zeta\rangle^4}{\langle 1,4\rangle\langle 2,1\rangle\langle 3,1\rangle\langle 4,1\rangle\langle 2,\zeta\rangle^2\langle 3,\zeta\rangle^2}\,. \tag{2.26}$$

By permutations between 1,2,3, and 4, there exist also eleven different subamplitudes.

Now it is ready to collect all the contributions from subamplitudes we have considered. By combining all the contributions, (2.19), (2.20), (2.21)(and corresponding five more terms), (2.23)(and eleven more terms), (2.24)(and twenty three more terms), (2.25)(and twenty three more terms), and (2.26)(and eleven more terms), the full amplitude consisting of eighty one($3^4 = 81$) terms can be simplified as

$$(-i)^4 \langle r, s\rangle^4 \prod_{j=1}^{4} \sum_{k(\neq j)=1}^{4} \frac{[j,k]\langle k,\zeta\rangle^2}{\langle j,k\rangle\langle j,\zeta\rangle^2} \tag{2.27}$$

which is an expression for the MHV amplitude [3] for scattering of four particles: two gravitons of helicity 2 and two particles labeled by r and s which are scalars $\overline{C}$ of helicity 0. From the results (2.15) and (2.27), one expects that the MHV amplitude for scattering of N gravitons can be extended by generalizing the summation and product index to N with overall factor $(-i)^N$.

What happens for the system of graviton plus gluons? We will describe them next.

2.3 MHV tree amplitude for both gluons and supergravitons

In this subsection, from the correlators between stress-energy tensor and current, we will describe MHV tree amplitudes for gravitons and gluons. Let us first consider the three-point amplitude.

Three point amplitude for one graviton and two gluons. The OPE between stress-energy tensor and the current takes the standard form [20, 22]

$$T(z_1)J^{r_2}(z_2) = \frac{J^{r_2}(z_2)}{z_{12}^2} + \frac{\partial J^{r_2}(z_2)}{z_{12}} + \cdots\,. \tag{2.28}$$

It is well-known that the current is a primary field of conformal dimension 1. From the OPE's (2.2) and (2.28), the nonzero three-point function is given by two contributions from the contraction of z_1 and z_2 and the contraction of z_1 and z_3:

$$\langle T(z_1)J^{r_2}(z_2)J^{r_3}(z_3)\rangle = k\delta^{r_2 r_3}\left(\frac{1}{z_{12}^2 z_{23}^2} - \frac{2}{z_{12}z_{23}^3} + (2\leftrightarrow 3)\right) \tag{2.29}$$

where the second term which contains $1/z_{23}^3$ is due to the correlation function $\langle \partial J^{r_2}(z_2)J^{r_3}(z_3)\rangle$ and there is a trivial three point function coming from the fact that $\langle T(z_1)J^{r_2}(z_2)\rangle = 0$:

$$\langle T(z_1)T(z_2)J^{r_3}(z_3)\rangle = 0\,.$$

Given the three-point function (2.29), how to construct the scattering amplitudes? Somehow the information for two gluons at z_2 and z_3 is encoded in (2.29).

Let us compute the amplitude using the above three-point correlation function (2.29). First, let us describe the case where the μ-derivative at 1^{st} particle acts on the wavefunction of 2^{nd} particle. We expect to have $f^{\dot a}$ for the wavefunction of supergraviton at z_1 and two ϕ's for the wavefunction of two gluons at z_2 and z_3 in the integrand. We introduce a factor (π_i^a/π_i^1) which is homogeneous of degree 0 in π_i(does not give any scale in π_i) and degree 1 in λ_i by recalling (2.4). This linear behavior is required by the vanishing of the sum of degrees of λ_i's. The vertex operator is given by $f_1^{\dot a}\frac{\partial}{\partial\mu_2^{\dot a}}\phi_2\phi_3$ together with correlator $\langle T(z_1)J^{r_2}(z_2)J^{r_3}(z_3)\rangle$ and prefactors $(\pi_1^a/\pi_1^1)(\pi_{2a}/\pi_2^1)$. This vertex operator scales as t^{-4} for 1^{st} particle and t^{-2} for 2^{nd} and 3^{rd} particles, as we expected. The subamplitude by inserting the wavefunctions (2.1) and (2.7) can be written as

$$\begin{aligned}\int d^4x d^8\theta dz_1 dz_2 dz_3\; &\left(\frac{\pi_1^a}{\pi_1^1}\right)\delta(\langle\lambda_1,\pi_1\rangle)\left(\frac{\lambda_1}{\pi_1}\right)^2 \widetilde{\pi}_1^{\dot a}\frac{\partial}{\partial\mu_2^{\dot a}}\exp\left(i\left(\frac{\pi_2}{\lambda_2}\right)[\mu_2,\widetilde{\pi}_2]\right)\times\\ &\times\left(\frac{\pi_{2a}}{\pi_2^1}\right)\delta(\langle\lambda_2,\pi_2\rangle)\left(\frac{\lambda_2}{\pi_2}\right)\exp\left(i\left(\frac{\pi_3}{\lambda_3}\right)[\mu_3,\widetilde{\pi}_3]\right)\times\\ &\times\delta(\langle\lambda_3,\pi_3\rangle)\left(\frac{\lambda_3}{\pi_3}\right)\exp\left(i\left(\frac{\pi_1}{\lambda_1}\right)[\mu_1,\widetilde{\pi}_1]\right)\times\\ &\times\langle T(z_1)J^{r_2}(z_2)J^{r_3}(z_3)\rangle\prod_{j=1}^{3}u\left(\left(\frac{\pi_j}{\lambda_j}\right)\psi_j\right).\end{aligned}$$

In the correlator (2.29), there are two contributions, but the correct term is related to a term $\frac{1}{z_{12}^2 z_{23}^2}$ since the amplitude for gluons(2^{nd} particle and 3^{rd} particle) should behave as $\frac{1}{z_{23}^2}$. The prefactor (π_i^a/π_i^1) is homogeneous of degree 0 in π and of degree 1 in λ. After differentiating the exponential with respect to μ, the factor $[1,2]$ for the spinors of negative chirality arises in this amplitude. The dependence on π_3^1 disappears because the factor $1/(\pi_3^1)^2$ is canceled out $(\pi_3^1)^2$ from the correlator. Using the second relation of (2.4), the factor z_{12} in the denominator of the correlator $\langle T(z_1)J^{r_2}(z_2)J^{r_3}(z_3)\rangle$, characterized by (2.29), is removed from the prefactor. As a result, the factor $\langle 1,2\rangle\langle 2,3\rangle^2$ for the spinors of positive chirality appears in the denominator of this amplitude. This amplitude can be simplified as

$$(-i)\frac{1}{\langle 2,3\rangle^2}\langle r,s\rangle^4\frac{[1,2]\langle 2,\zeta\rangle^2}{\langle 1,2\rangle\langle 1\,,\zeta\rangle^2}\,. \tag{2.30}$$

Definitely, the sum of degrees of λ_1 and λ_2 is equal to zero. The factor $1/\langle 2,3\rangle^2$ implies the MHV tree amplitude for two gluons(omitting the group theory factor).

Moreover, one can compute the case where the μ-derivative at 1^{st} particle acts on the wavefunction of 3^{rd} particle instead of 2^{nd} particle. The corresponding correlator for this case is given by $\frac{1}{z_{13}^2 z_{32}^2}$ from (2.29). Therefore, the full amplitude coming from these two contributions becomes, by summing over 2^{nd} and 3^{rd} particles,

$$(-i)\frac{1}{\langle 2,3\rangle^2}\langle r,s\rangle^4 \sum_{k=2}^{3} \frac{[1,k]\langle k,\zeta\rangle^2}{\langle 1,k\rangle\langle 1,\zeta\rangle^2} \tag{2.31}$$

which is an expression for the MHV tree amplitude [3] for scattering of three particles: a graviton of helicity 2 characterized by the 1^{st} particle and two particles which are gauge bosons(labeled by r and s) of helicity -1.

So far, we have constructed the vertex operator at each particle and computed the amplitudes by calculating the integrals. The behavior of prefactor is linear and is determined by the requirement that the sum of degrees of λ_i's vanishes. Now we continue to the higher correlation function next.

Four point amplitude for one graviton and three gluons. By considering the contractions between z_1 and z_2, z_3, z_4, the four-point function can be obtained and is given by

$$\langle T(z_1)J^{r_2}(z_2)J^{r_3}(z_3)J^{r_4}(z_4)\rangle = \frac{f^{r_2r_3r_4}}{z_{12}z_{23}z_{34}z_{42}}\left(\frac{1}{z_{12}}-\frac{1}{z_{23}}+\frac{1}{z_{42}}\right)+(2\leftrightarrow 3)+(2\leftrightarrow 4)\,. \tag{2.32}$$

Note that the overall factor of the first term has a dependence on $\frac{1}{z_{23}z_{34}z_{42}}$ which can be seen from three-point correlator of the current (2.3) and only this first term is relevant to our scattering amplitude.

The scattering amplitude where the twistor derivative with respect to μ at 1^{st} particle acts on the wavefunction of 2^{nd} particle which is simply an extension of (2.30) for three gluons. The vertex operator is given by $f_1^{\dot{a}}\frac{\partial}{\partial\mu_2^{\dot{a}}}\phi_2\phi_3\phi_4$ as well as the correlator (2.32) and prefactors. Then the subamplitude, by adding the 4^{th} particle, can be written as

$$\begin{aligned}
&\left(\frac{\pi_1^a}{\pi_1^1}\right)\left(\frac{\pi_{2a}}{\pi_2^1}\right)\left(\frac{\lambda_1}{\pi_1}\right)^2 \widetilde{\pi}_1^{\dot{a}}\frac{\partial}{\partial\mu_2^{\dot{a}}}\exp\left(i\left(\frac{\pi_2}{\lambda_2}\right)[\mu_2,\widetilde{\pi}_2]\right)\left(\frac{\lambda_2}{\pi_2}\right)\exp\left(i\left(\frac{\pi_3}{\lambda_3}\right)[\mu_3,\widetilde{\pi}_3]\right)\times\\
&\times\left(\frac{\lambda_3}{\pi_3}\right)\exp\left(i\left(\frac{\pi_4}{\lambda_4}\right)[\mu_4,\widetilde{\pi}_4]\right)\left(\frac{\lambda_4}{\pi_4}\right)\exp\left(i\left(\frac{\pi_1}{\lambda_1}\right)[\mu_1,\widetilde{\pi}_1]\right)\times\\
&\times\langle T(z_1)\prod_{k=2}^{4}J^{r_k}(z_k)\rangle\prod_{j=1}^{4}\delta(\langle\lambda_j,\pi_j\rangle)u\left(\left(\frac{\pi_j}{\lambda_j}\right)\psi_j\right)
\end{aligned}$$

where we did not write down the integrals explicitly for simplicity. In the correlator (2.32), the only first term is relevant to our discussion because the amplitude for three gluons should behave like $\frac{1}{z_{23}z_{34}z_{42}}$. The factor $[1,2]$ for the spinors of negative chirality arises in this amplitude by differentiating the exponential with respect to μ_2. Using (2.4), the factor z_{12} in the denominator of the correlator $\langle T(z_1)J^{r_2}(z_2)J^{r_3}(z_3)J^{r_4}(z_4)\rangle$, characterized by (2.32), is canceled by the prefactors. As a result, the factor $\langle 1,2\rangle\langle 2,3\rangle\langle 3,4\rangle\langle 4,2\rangle$ for the spinors of positive chirality appears in the denominator of this amplitude. Then the amplitude can be written as

$$(-i)\frac{1}{\langle 2,3\rangle\langle 3,4\rangle\langle 4,2\rangle}\langle r,s\rangle^4\frac{[1,2]\langle 2,\zeta\rangle^2}{\langle 1,2\rangle\langle 1,\zeta\rangle^2}\,.$$

One can proceed the other two cases where the role of 2^{nd} particle is replaced by 3^{rd} and 4^{th} particles and summing over these contributions, one arrives at the full amplitude given by

$$(-i)\frac{1}{\langle 2,3\rangle\langle 3,4\rangle\langle 4,2\rangle}\langle r,s\rangle^4\sum_{k=2}^{4}\frac{[1,k]\langle k,\zeta\rangle^2}{\langle 1,k\rangle\langle 1,\zeta\rangle^2}\,. \tag{2.33}$$

This is the MHV tree amplitude [3] for scattering of four particles: a graviton of helicity 2, a gauge boson of helicity 1 and two particles which are gauge bosons of helicity -1.

One can easily see the generalization of this scattering amplitude to N external particles consisting of one graviton and $(N-1)$ gluons. The result is obtained by summing over k to N in (2.33) and the prefactor has the general expression for $(N-1)$ gluons:$\prod_{i=2}^{N}\frac{1}{\langle i,i+1\rangle}$.

Four point amplitude for two gravitons and two gluons. Let us consider the cases where there are two gravitons. There exists a four-point function

$$\langle T(z_1)T(z_2)T(z_3)J^{r_4}(z_4)\rangle = 0$$

which can be obtained from the vanishing of correlation functions $\langle T(z_1)T(z_2)J^{r_3}(z_3)\rangle = 0$ and $\langle T(z_1)J^{r_2}(z_2)\rangle = 0$ and the following four-point function has rather complicated expression

$$\begin{aligned}\langle T(z_1)T(z_2)J^{r_3}(z_3)J^{r_4}(z_4)\rangle = &\frac{ck\delta^{r_3r_4}/2}{z_{12}^4z_{34}^2}+k\delta^{r_3r_4}\left(\frac{2}{z_{12}^2}+\frac{1}{z_{13}^2}+\frac{1}{z_{14}^2}\right)\times\\ &\times\left(\frac{1}{z_{23}^2z_{34}^2}-\frac{2}{z_{23}z_{34}^3}+(3\leftrightarrow 4)\right)+\\ &+\text{other singular terms}\end{aligned} \tag{2.34}$$

where the first term comes from the disconnected piece $\langle T(z_1)T(z_2)\rangle\langle J^{r_3}(z_3)J^{r_4}(z_4)\rangle$.[5] The term of $1/z_{34}^2$ in the second term provides the correlator for two currents and is relevant to our discussion.

Let us consider the subamplitude where the μ-derivative of vertex operator at 1^{st} particle acts on the wavefunction of 2^{nd} particle and the μ-derivative of vertex operator at

[5]It is easy to see that the other singular terms are given by the following correlation functions $\frac{1}{z_{12}}\langle\partial T(z_2)J^{r_3}(z_3)J^{r_4}(z_4)\rangle+\frac{1}{z_{13}}\langle T(z_2)\partial J^{r_3}(z_3)J^{r_4}(z_4)\rangle+\frac{1}{z_{14}}\langle T(z_2)J^{r_3}(z_3)\partial J^{r_4}(z_4)\rangle$.

2^{nd} particle acts on the wavefunction of 1^{st} particle. In this case, the prefactors coming from the two gravitons are the same as the one we have discussed in the previous subsection around (2.19). There is no π_i^1 dependence in the amplitude. We expect to have two ϕ's from two gluons. The integrand we are interested in is

$$\left(\frac{\pi_1^a \pi_{1b}}{\pi_1^1 \pi_1^1}\right)\left(\frac{\pi_2^b \pi_{2a}}{\pi_2^1 \pi_2^1}\right) H(z_k, \pi_j, \lambda_j, \psi_j)\left(\frac{\lambda_1}{\pi_1}\right)^2 \widetilde{\pi}_1^{\dot{a}} \frac{\partial}{\partial \mu_2^{\dot{a}}} \exp\left(i\left(\frac{\pi_2}{\lambda_2}\right)[\mu_2, \widetilde{\pi}_2]\right) \times$$
$$\times \left(\frac{\lambda_2}{\pi_2}\right)^2 \widetilde{\pi}_2^{\dot{b}} \frac{\partial}{\partial \mu_1^{\dot{b}}} \exp\left(i\left(\frac{\pi_1}{\lambda_1}\right)[\mu_1, \widetilde{\pi}_1]\right)\left(\frac{\lambda_3}{\pi_3}\right) \exp\left(i\left(\frac{\pi_3}{\lambda_3}\right)[\mu_3, \widetilde{\pi}_3]\right) \times$$
$$\times \left(\frac{\lambda_4}{\pi_4}\right) \exp\left(i\left(\frac{\pi_4}{\lambda_4}\right)[\mu_4, \widetilde{\pi}_4]\right)$$

where we introduce a new notation which will appear several times

$$H(z_k, \pi_j, \lambda_j, \psi_j) \equiv \langle T(z_1) T(z_2) J^{r_3}(z_3) J^{r_4}(z_4)\rangle \prod_{j=1}^{4} \delta(\langle \lambda_j, \pi_j\rangle) u\left(\left(\frac{\pi_j}{\lambda_j}\right)\psi_j\right). \tag{2.35}$$

The relevant term in the correlator (2.34) is $\frac{1}{z_{12}^4 z_{34}^2}$ because in this case the amplitude for gluons(3^{rd} particle and 4^{th} particle) behaves correctly. After simplifying the above(the prefactors kill z_{12}^2) and extracting the first term for linear k in the correlator (2.34), the subamplitude can be written

$$(-i)^2 \frac{1}{\langle 3,4\rangle^2} \langle r,s\rangle^4 \frac{[1,2][2,1]}{\langle 1,2\rangle\langle 2,1\rangle}\,. \tag{2.36}$$

The factor $1/\langle 3,4\rangle^2$ implies the MHV tree amplitude for two gluons.

Let us consider the subamplitude where the μ-derivative of vertex operator at 1^{st} particle acts on the wavefunction of 2^{nd} particle and the μ-derivative of vertex operator at 2^{nd} particle acts on the wavefunction of 3^{rd} particle. The prefactor in 2^{nd} particle is quadratic in π_2^a because the spinor indices are contracted with the one in 1^{st} particle and the one in 3^{rd} particle. Then with appropriate prefactors which will remove some factors in the correlator (2.34) where the relevant term is $\frac{1}{z_{12}^2 z_{23}^2 z_{34}^2}$, the subamplitude with (2.35) is

$$\left(\frac{\pi_1^a}{\pi_1^1}\right)\left(\frac{\pi_2^b \pi_{2a}}{\pi_2^1 \pi_2^1}\right)\left(\frac{\pi_{3b}}{\pi_3^1}\right) H(z_k, \pi_j, \lambda_j, \psi_j)\left(\frac{\lambda_1}{\pi_1}\right)^2 \widetilde{\pi}_1^{\dot{a}} \frac{\partial}{\partial \mu_2^{\dot{a}}} \exp\left(i\left(\frac{\pi_2}{\lambda_2}\right)[\mu_2, \widetilde{\pi}_2]\right) \times$$
$$\times \left(\frac{\lambda_2}{\pi_2}\right)^2 \widetilde{\pi}_2^{\dot{b}} \frac{\partial}{\partial \mu_3^{\dot{b}}} \exp\left(i\left(\frac{\pi_3}{\lambda_3}\right)[\mu_3, \widetilde{\pi}_3]\right)\left(\frac{\lambda_3}{\pi_3}\right) \exp\left(i\left(\frac{\pi_4}{\lambda_4}\right)[\mu_4, \widetilde{\pi}_4]\right) \times$$
$$\times \left(\frac{\lambda_4}{\pi_4}\right) \exp\left(i\left(\frac{\pi_1}{\lambda_1}\right)[\mu_1, \widetilde{\pi}_1]\right).$$

It can be easily checked that the degree of λ_1 is 2 while the degree of λ_3 is -2 but the sum of degrees is equal to zero. There is also other contribution by taking the μ-derivative of vertex operator at 2^{nd} particle which acts on the wavefunction of 4^{th} particle instead of 3^{rd} particle($3 \leftrightarrow 4$). In this case, we have to use the corresponding relevant term in (2.34).

Then the total subamplitude coming from two contributions can be written as

$$(-i)^2 \frac{1}{\langle 3,4\rangle^2} \langle r,s\rangle^4 \left(\frac{[1,2][2,3]\langle 3,\zeta\rangle^2}{\langle 1,2\rangle\langle 2,3\rangle\langle 1,\zeta\rangle^2} + (3 \leftrightarrow 4) \right). \tag{2.37}$$

Obviously, the sum of degrees for λ_1 and λ_3 is zero.[6]

Let us consider the subamplitude where the μ-derivatives of vertex operator at 1^{st} particle and 2^{nd} particle acts on the wavefunction of 3^{rd} particle simultaneously. The prefactor for 3^{rd} particle is quadratic and the prefactors for other particle are linear. The result is

$$\begin{aligned} &\left(\frac{\pi_1^a}{\pi_1^1}\right)\left(\frac{\pi_2^b}{\pi_2^1}\right)\left(\frac{\pi_{3a}\pi_{3b}}{\pi_3^1\pi_3^1}\right) H(z_k,\pi_j,\lambda_j,\psi_j)\left(\frac{\lambda_1}{\pi_1}\right)^2\left(\frac{\lambda_2}{\pi_2}\right)^2 \widetilde{\pi}_1^{\dot a}\widetilde{\pi}_2^{\dot b} \times \\ &\times \frac{\partial}{\partial\mu_3^{\dot a}}\frac{\partial}{\partial\mu_3^{\dot b}} \exp\left(i\left(\frac{\pi_3}{\lambda_3}\right)[\mu_3,\widetilde{\pi}_3]\right) \exp\left(i\left(\frac{\pi_4}{\lambda_4}\right)[\mu_4,\widetilde{\pi}_4]\right) \times \\ &\times \left(\frac{\lambda_3}{\pi_3}\right) \exp\left(i\left(\frac{\pi_1}{\lambda_1}\right)[\mu_1,\widetilde{\pi}_1]\right)\left(\frac{\lambda_4}{\pi_4}\right)\exp\left(i\left(\frac{\pi_2}{\lambda_2}\right)[\mu_2,\widetilde{\pi}_2]\right). \end{aligned}$$

For 3^{rd} particle, the prefactor is quadratic and there are two μ_3-derivatives. Then one can easily check that the degree of λ_3 is equal to 4. On the other hand, for 4^{th} particle, the factor $1/(\pi_4^1)^2$ coming from both the scale above and delta function is canceled out $(\pi_4^1)^2$ in $1/z_{34}^2$. For 1^{st} and 2^{nd} particles, the scales $1/(\pi_1^1)^2$ and $1/(\pi_2^1)^2$ remain in the amplitude. Finally, the sum of degrees for λ_i's is equal to zero. Note that the relevant term in the correlator (2.34) is $\frac{1}{z_{13}^2 z_{23}^2 z_{34}^2}$. In this case, the subamplitude, by adding other contribution(by changing the role of 3^{rd} particle and 4^{th} particle), can be summarized

$$(-i)^2 \frac{1}{\langle 3,4\rangle^2} \langle r,s\rangle^4 \left(\frac{[1,3][2,3]\langle 3,\zeta\rangle^4}{\langle 1,3\rangle\langle 2,3\rangle\langle 1,\zeta\rangle^2\langle 2,\zeta\rangle^2} + (3 \leftrightarrow 4) \right). \tag{2.38}$$

Finally, let us describe the subamplitude where the μ-derivative of vertex operator at 1^{st} particle acts on the wavefunction of 3^{rd} particle and the μ-derivative of vertex operator at 2^{nd} particle acts on the wavefunction of 4^{th} particle. All the prefactors in this case are linear in π_i^a for each i^{th} particle. The subamplitude with (2.35) is given by

$$\begin{aligned} &\left(\frac{\pi_1^a}{\pi_1^1}\right)\left(\frac{\pi_2^b}{\pi_2^1}\right)\left(\frac{\pi_{3a}}{\pi_3^1}\right)\left(\frac{\pi_{4b}}{\pi_4^1}\right) H(z_k,\pi_j,\lambda_j,\psi_j)\left(\frac{\lambda_1}{\pi_1}\right)^2 \widetilde{\pi}_1^{\dot a}\frac{\partial}{\partial\mu_3^{\dot a}} \exp\left(i\left(\frac{\pi_3}{\lambda_3}\right)[\mu_3,\widetilde{\pi}_3]\right) \times \\ &\times \left(\frac{\lambda_2}{\pi_2}\right)^2 \widetilde{\pi}_2^{\dot b}\frac{\partial}{\partial\mu_4^{\dot b}} \exp\left(i\left(\frac{\pi_4}{\lambda_4}\right)[\mu_4,\widetilde{\pi}_4]\right)\left(\frac{\lambda_3}{\pi_3}\right)\exp\left(i\left(\frac{\pi_1}{\lambda_1}\right)[\mu_1,\widetilde{\pi}_1]\right) \times \\ &\times \left(\frac{\lambda_4}{\pi_4}\right)\exp\left(i\left(\frac{\pi_2}{\lambda_2}\right)[\mu_2,\widetilde{\pi}_2]\right). \end{aligned}$$

[6] For the subamplitude where the μ-derivative of vertex operator at 1^{st} particle acts on the wavefunction of 3^{rd} particle and the μ-derivative of vertex operator at 2^{nd} particle acts on the wavefunction of 1^{st} particle, the prefactor of 1^{st} particle is quadratic in π_1^a. The subamplitude is given by similarly. By taking account of other case which changes the role of 3^{rd} particle($3 \leftrightarrow 4$), one gets $(-i)^2\frac{1}{\langle 3,4\rangle^2}\langle r,s\rangle^4\left(\frac{[1,3][2,1]\langle 3,\zeta\rangle^2}{\langle 1,3\rangle\langle 2,1\rangle\langle 2,\zeta\rangle^2} + (3\leftrightarrow 4)\right)$. Actually this can be obtained from the result (2.37) by exchanging the role of 1^{st} particle and 2^{nd} particle($1 \leftrightarrow 2$). This is the reason why the coefficient of $\frac{1}{z_{12}^2 z_{23}^2 z_{34}^2}$ in (2.34) is twice as large as those of other terms.

As we have done before, in this case, the dependence on π_i^1 where $i = 1, 2, 3, 4$ in this amplitude occurs. For 1^{st} and 2^{nd} particles, the scales $1/(\pi_1^1)^2$ and $1/(\pi_2^1)^2$ remain in the amplitude. On the other hand, for 3^{rd} and 4^{th} particles, the factors $(\pi_3^1)^2$ and $(\pi_4^1)^2$ appear. The sum of degrees of λ_i's is zero. The relevant term in the correlator (2.34) is $\frac{1}{z_{13}^2 z_{24}^2 z_{34}^2}$ because in this case the amplitude for gluons(3^{rd} particle and 4^{th} particle) behaves correctly. After we take into account of other case where the role of 3^{rd} particle and 4^{th} particle is interchanged($3 \leftrightarrow 4$), the final expression is given by

$$(-i)^2 \frac{1}{\langle 3,4\rangle^2} \langle r,s\rangle^4 \left(\frac{[1,3][2,4]\langle 3,\zeta\rangle^2 \langle 4,\zeta\rangle^2}{\langle 1,3\rangle\langle 2,4\rangle\langle 1,\zeta\rangle^2 \langle 2,\zeta\rangle^2} + (3 \leftrightarrow 4) \right). \tag{2.39}$$

From the contributions (2.36), (2.37)(and two other terms), (2.38) and (2.39), the full amplitude($3^2 = 9$) by rearrangement is given by

$$(-i)^4 \frac{1}{\langle 3,4\rangle^2} \langle r,s\rangle^4 \prod_{j=1}^{2} \sum_{k(\neq j)=1}^{4} \frac{[j,k]\langle k,\zeta\rangle^2}{\langle j,k\rangle\langle j,\zeta\rangle^2}. \tag{2.40}$$

which is an expression for the MHV tree amplitude [3] for scattering of four particles: two gravitons of helicity 2 and two gluons labeled by r and s of helicity -1. The generalization of this scattering amplitude to N external particles consisting of two gravitons and $(N-2)$ gluons can be obtained by summing over k to N in (2.40) and the prefactor has the general expression for $(N-2)$ gluons.

3. Discussions

We have obtained three-point and four-point amplitudes for gravitons (2.15) and (2.27), three-point and four-point amplitudes for gravitons and gluons (2.31), (2.33) and (2.40). These results agree with the description [3] from open string version. Can we generalize for arbitrary N-point($N \geq 5$) amplitudes as we mentioned before? It is rather obvious that from the open string version of twistor-string theory, the MHV tree scattering amplitudes is generalized to arbitrary N-point amplitudes, starting from the scattering amplitude between any two different particles.

Now we will describe the five-point amplitudes very briefly in the context of topological B-model. Let us first consider the amplitude for five gravitons and we will add the gluons later. The latter is easier to analyze than the former.

Five-point amplitude for five gravitons. Now let us consider the five-point function. One can find this from the OPE's between stress-energy tensor.[7] Using the relation,[8] the

[7] For example, by reading off the contractions between z_1 and z_k where $k = 1, \ldots, 4$, the quadratic terms of c in this correlator $\langle \prod_{k=1}^{5} T(z_k) \rangle$ are given by $\frac{c^2}{2} \frac{1}{z_{12}^4 z_{34}^2 z_{45}^2 z_{53}^2} \left[1 + x^2 y^2 + \frac{x^2(x-y)^2}{(1-x)^4} + \frac{y^2(x-y)^2}{(1-y)^4} \right]$ where we introduced the two variables $x \equiv \frac{z_{12} z_{34}}{z_{13} z_{24}}, y \equiv \frac{z_{12} z_{35}}{z_{13} z_{25}}$.

[8] It is easy to check that $\frac{1}{z_{12}^4 z_{34}^2 z_{45}^2 z_{53}^2} = \frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{45}^2 z_{51}^2} \left(\frac{1-y}{y} \right)^2$.

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five-point function[9] can be written as

$$\frac{2c}{z_{12}^2 z_{23}^2 z_{34}^2 z_{45}^2 z_{51}^2} + \text{other singular terms}\,.$$

For the case of five gravitons, there exist $4^5 = 1024$ terms for the scattering amplitudes. When we take the contraction of spinor indices between $\widetilde{\pi}$ and $\frac{\partial}{\partial \mu}$, there are following different inner products $[1,3], [2,4], [3,5], [4,1], [5,2]$ where the corresponding factors for the spinors of positive chirality $\langle 1,3\rangle, \langle 2,4\rangle, \langle 3,5\rangle, \langle 4,1\rangle, \langle 5,2\rangle$ are not present in the correlator $\prod_{k=1}^{5}\langle T(z_k)\rangle$. Recall that for four graviton scattering amplitude, there exist two cases of $\langle i, i+2\rangle$ where $i = 1, 2$ which do not exist in the correlator of stress-energy tensor. As we have seen in (2.24), (2.25), and (2.26) for four-point function, we should consider the following factors for five-point function:

$$\left(\frac{\pi_1^a \pi_{3a}}{\pi_1^1 \pi_3^1}\right)^{-1}, \quad \left(\frac{\pi_2^a \pi_{4a}}{\pi_2^1 \pi_4^1}\right)^{-1}, \quad \left(\frac{\pi_3^a \pi_{5a}}{\pi_3^1 \pi_5^1}\right)^{-1}, \quad \left(\frac{\pi_4^a \pi_{1a}}{\pi_4^1 \pi_1^1}\right)^{-1}, \quad \left(\frac{\pi_5^a \pi_{2a}}{\pi_5^1 \pi_2^1}\right)^{-1}$$

as a prefactor of a vertex operator at each external particle. Of course, we can consider part of these and the remaining factors can be obtained by permutations between particles, as we have seen in four-particle scattering amplitude. Part of subamplitudes contain the factors

$$\frac{\langle i,\zeta\rangle^8}{\langle j,\zeta\rangle^2\langle k,\zeta\rangle^2\langle l,\zeta\rangle^2\langle m,\zeta\rangle^2}, \qquad \frac{\langle i,\zeta\rangle^6}{\langle j,\zeta\rangle^2\langle k,\zeta\rangle^2\langle l,\zeta\rangle^2}, \qquad \frac{\langle i,\zeta\rangle^4\langle j,\zeta\rangle^2}{\langle k,\zeta\rangle^2\langle l,\zeta\rangle^2\langle m,\zeta\rangle^2}$$

where $i, j, k, l, m = 1, \ldots, 5$, compared with four-point amplitude. They are homogeneous of degree zero in sum of each λ_i although each λ_i has different degrees in each case. We expect, by counting the independent subamplitudes with correct multiplicities which will be very tedious, that we can write down the MHV tree amplitude for scattering of five gravitons will be coincident with the result of [3] eventually.

Five-point amplitude for one graviton and four gluons. As we have done in (2.32) before, the expression for the correlator $\langle T(z_1)\prod_{k=2}^{5} J^{r_k}(z_k)\rangle$ can be obtained from the basic OPEs (2.2) and (2.28). One can think of one of the terms in this correlator: The factor $\frac{1}{z_{12}^2}$ comes from the contraction between z_1 and z_2 and there is a four-point correlator $\langle \prod_{k=2}^{5} J^{r_k}(z_k)\rangle$ whose explicit form was given in the footnote 1 already. The relevant term in the five-point correlator will contain $\frac{1}{z_{12}^2 z_{23} z_{34} z_{45} z_{53}}$. One expects that the subamplitude for this case looks like $(-i)\left(\prod_{k=2}^{5}\frac{1}{\langle k,k+1\rangle}\right)\langle r,s\rangle^4 \frac{[1,2]\langle 2,\zeta\rangle^2}{\langle 1,2\rangle\langle 1,\zeta\rangle^2}$, as we already mentioned in the last paragraph after (2.33). Together with other three cases, the full amplitude can be written similar to (2.33) without any difficulty.

[9] The linear terms in c of $\langle\prod_{k=1}^{5} T(z_k)\rangle$ can be summarized by $\frac{2c}{z_{12}^4 z_{34}^2 z_{45}^2 z_{53}^2}\frac{1}{(1-x)^2(1-y)^2}F(x,y)$ where we introduce $F(x,y) \equiv 2x^4(1-y+y^2)+2y^2(1-y+y^2)-x^3(2+y+y^2+2y^3)-xy(2+y+y^2+2y^3)+x^2(2-y+6y^2-y^3+2y^4)$.

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Five-point amplitude for two gravitons and three gluons. One can see the term $\frac{1}{z_{12}^2}\langle T(z_2)\prod_{k=3}^5 J^{r_k}(z_k)\rangle$ by contracting of two stress-energy tensor, in the correlator $\langle T(z_1)T(z_2)\prod_{k=3}^5 J^{r_k}(z_k)\rangle$. We have already seen the four-point function in (2.32). Then it is easy to check that the subamplitude for this case corresponds to $\frac{1}{z_{12}^2 z_{23}^2 z_{34}z_{45}z_{53}}$. Clearly, the factor $\frac{1}{z_{34}z_{45}z_{53}}$ will enter into the MHV tree amplitude for three gluons. We will obtain the final expression of this amplitude, as mentioned in (2.40).

Five-point amplitude for three gravitons and two gluons. Since the contraction z_1 and z_2 gives a term $\frac{1}{z_{12}^2}T(z_2)$, this five-point correlator contains $\frac{1}{z_{12}^2}\langle T(z_2)T(z_3)J^{r_4}(z_4)\times J^{r_5}(z_5)\rangle$. According to (2.34), one can read off a term like $\frac{1}{z_{23}^2 z_{34}^2 z_{45}^2}$ from those four-point function. Therefore, one can see the total factor $\frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{45}^2}$ in the correlator. The factor $1/z_{45}^2$ will provide the MHV tree amplitude for two gluons at z_4 and z_5 and the others with appropriate prefactors will give us other factor with gravitons.

Five-point amplitude for four gravitons and one gluon. Since the correlator between stress-energy tensor and the current vanishes, the five-point correlator

$$\left\langle \left(\prod_{k=1}^4 T(z_k)\right) J^{r_5}(z_5)\right\rangle = 0\,.$$

There is no contribution in the amplitude.

In this way, one can construct higher N-point MHV tree amplitudes($N \geq 5$) which will coincide with the result of [3].

One can think of conformal dimension 3 operator, represented by third order Casimir operator of the underlying representation of Lie algebra SU(N), taking the form $W(z) = d_{r_1r_2r_3}J^{r_1}J^{r_2}J^{r_3}(z)$ [22]: triple product of current. Here $d_{r_1r_2r_3} = \mathrm{Tr}(\{T^{r_1},T^{r_2}\}T^{r_3})$ is a completely symmetric traceless tensor and its indices are contracted with the color indices of the current. Then it is easy to see this $W(z)$ has a conformal dimension 3 under the stress-energy tensor $T(z)$. The four-point function of $W(z)$ [23] can be obtained from their basic OPE's $W(z_1)W(z_2)$ and $T(z_1)W(z_2)$ and is given by[10] $\langle\prod_{k=1}^4 W(z_k)\rangle = G(x)z_{13}^{-6}z_{24}^{-6}$ where $G(x) = \cdots + 2c\left[\frac{1}{x^3}+\frac{1}{(1-x)^3}\right]+\cdots$ and x is defined as (2.16). The complete expression for four-point function is already in [23]. Then it is easy to see that the four-point function contains a factor $\frac{1}{z_{12}^3 z_{23}^3 z_{34}^3 z_{41}^3}$ as well as other singular terms. Recall that $W(z)$ has a conformal dimension 3. This shares common feature with the four-point functions of dimension 1(current) in the footnote 1 and dimension 2(stress-energy tensor) (2.17). It would be interesting to see how this dimension 3 operator and its correlators occur in the scattering amplitudes in context of the topological B-model.

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[10]It can be checked that the two-point and three-point functions are given by $\langle W(z_1)W(z_2)\rangle = \frac{c/3}{z_{12}^6}$ and $\langle W(z_1)W(z_2)W(z_3)\rangle = 0$.

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